

SPECTRAL THEOREM FOR UNBOUNDED NORMAL OPERATORS IN QUATERNIONIC HILBERT SPACES

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ABSTRACT. In this article, we prove the following spectral theorem for right linear normal operators (need not to be bounded) in quaternionic Hilbert spaces: Let T be an unbounded right quaternionic linear normal operator in a quaternionic Hilbert space H with domain $\mathcal{D}(T)$, a right linear subspace of H and fix a unit imaginary quaternion, say m . Then there exists a Hilbert basis \mathcal{N} of H and a unique quaternionic spectral measure F on the σ -algebra of \mathbb{C}_m^+ (upper half plane of the slice complex plane \mathbb{C}_m) associated to T such that

$$\langle x|Ty \rangle = \int_{\sigma_S(T) \cap \mathbb{C}_m^+} \lambda \, dF_{x,y}(\lambda), \quad \text{for all } y \in \mathcal{D}(T), \, x \in H,$$

where $F_{x,y}$ is a quaternion valued measure on the σ -algebra of \mathbb{C}_m^+ , for any $x, y \in H$ and $\sigma_S(T)$ is the spherical spectrum of T . Here the representation of T is established with respect to the Hilbert basis \mathcal{N} .

To prove this result, we reduce the problem to the complex case and obtain the result by using the classical result.

1. INTRODUCTION AND PRELIMINARIES

Traditionally quantum mechanics has been done in complex Hilbert spaces which is referred as classical quantum mechanics. In classical quantum mechanics the spectral theorem for unbounded closed operators gives a one to one correspondence between observables of physical system and projection valued measures in a complex Hilbert space [11]. Since, the observables of a physical system are projection valued functions on the Borel sigma algebra of the field of real numbers, the Hilbert space must be defined on a associative division algebra which has reals as subfield [14]. Since quaternions is one such associative division algebra, the next interesting topic is Hilbert spaces over quaternions.

The study of the quaternionic quantum mechanics (QQM) was initiated by Birkhoff and von Neumann [5], in which the motivation of the spectral

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theorem for quaternionic operator was given (see section III of [5], for details). A series of papers indicates a significant development in the theory of QQM [1, 6, 7, 8]. To see the connection between QQM and operators in quaternionic Hilbert space, for instance, the quaternionic version of the Stone's theorem is a bridge between the Schrödinger equation in QQM and one parameter family of unitary group of quaternionic operators. The spectral theorem for quaternionic anti Hermitian, unitary operators plays a vital role in QQM [6, 8].

The spectral theorem that deals with the integral representation of a quaternionic normal operator is given in [15]. The author proved the existence of spectral measure through the symplectic image and as a consequence, he obtained the Cartesian decomposition of a normal operator on a quaternionic Hilbert space. An approach using real Banach algebra techniques, S.H. Kulkarni and Sushama Agarwal in [2] proved the spectral theorem for normal operators on real Hilbert spaces and deduced the quaternionic version from this. These two results does not use the concept of the spherical spectrum.

The spectral theorem for quaternionic unitary operators is proved in [14]. The same result was obtained by Alpay et al. using the notion of spherical spectrum and the quaternionic version of the Herglotz theorem [3]. Later this result was generalized to the case of unbounded normal operators by Alpay et al. in [4].

In this article, we prove the spectral theorem for unbounded quaternionic normal operator by associating a complex normal operator to the given operator. Our approach to prove integral representation for unbounded quaternionic normal operator is completely different from the one used in [4]. First we reduce the problem to the complex case by decomposing quaternionic Hilbert space into the direct sum of two slice Hilbert spaces. Then, we lift this result to the quaternionic case (See Theorem 2.10 of this article).

We organize this article in two sections. In the first section we fix some of the notations, recall some basis properties of the ring of quaternions and definitions.

In the second section we present some results related to unbounded quaternionic operators and prove the spectral theorem.

Quaternions: Let i, j, k be three mutually orthogonal axes, which satisfies $i^2 = j^2 = k^2 = -1 = i \cdot j \cdot k$. Let $\mathbb{H} = \{q = q_0 + q_1i + q_2j + q_3k : q_l \in \mathbb{R}, l = 0, 1, 2, 3\}$ denotes the division ring (skew field) of all real quaternions. The conjugate of q is $\bar{q} = q_0 - q_1i - q_2j - q_3k$. The real part of \mathbb{H} is denoted by $\text{Re}(\mathbb{H}) = \{q \in \mathbb{H} : q = \bar{q}\}$ and the imaginary part of \mathbb{H} is denoted by $\text{Im}(\mathbb{H}) = \{q \in \mathbb{H} : q = -\bar{q}\}$. The set $\mathbb{S} := \{q \in \text{Im}(\mathbb{H}) : |q| = 1\}$ is the unit sphere in $\text{Im}(\mathbb{H})$. For each $m \in \mathbb{S}$, define the slice complex plane, $\mathbb{C}_m := \{\alpha + m\beta : \alpha, \beta \in \mathbb{R}\}$, which is real sub algebra of \mathbb{H} .

For every $m \in \mathbb{S}$, the real algebra \mathbb{C}_m is isomorphic to \mathbb{C} through the map $\alpha + m\beta \mapsto \alpha + i\beta$. So, all the results on complex Hilbert spaces holds true on \mathbb{C}_m - Hilbert spaces.

Here we recall some of the elementary properties of quaternions (See [9] for details).

- (1) For $p, q \in \mathbb{H}$, $|p \cdot q| = |p| \cdot |q|$ and $|\bar{p}| = |p|$
- (2) Define $p \sim q$ if and only if $p = s^{-1}qs$, for some $0 \neq s \in \mathbb{H}$. Then \sim is an equivalence relation on H
- (3) The equivalence class of $q \in \mathbb{H}$ is $[q] := \{p \in \mathbb{H} : p \sim q\}$, equivalently, $[q] = \{p \in \mathbb{H} : \text{Re}(p) = \text{Re}(q) \text{ and } |\text{Im}(p)| = |\text{Im}(q)|\}$
- (4) For $m \neq \pm n \in \mathbb{S}$, we can see, $\mathbb{C}_m \cap \mathbb{C}_n = \mathbb{R}$ and $\mathbb{H} = \bigcup_{m \in \mathbb{S}} \mathbb{C}_m$
- (5) Let $q \in \mathbb{H}$. Then, $q \in \mathbb{C}_m$ if and only if $q \cdot \lambda = \lambda \cdot q$, for every $\lambda \in \mathbb{C}_m$

Let H be a right \mathbb{H} - module with a map $\langle \cdot | \cdot \rangle : H \times H \rightarrow \mathbb{H}$ satisfying the following three properties:

- (1) If $u \in H$, then $\langle u | u \rangle \geq 0$ and $\langle u | u \rangle = 0$ if and only if $u = 0$
- (2) $\langle u | vp + wq \rangle = \langle u | v \rangle p + \langle u | w \rangle q$, if $u, v, w \in H$ and $p, q \in \mathbb{H}$
- (3) $\langle u | v \rangle = \overline{\langle v | u \rangle}$.

Define $\|u\| = \sqrt{\langle u | u \rangle}$, for every $u \in H$. Then $\|\cdot\|$ is a norm on H . If the normed space $(H, \|\cdot\|)$ is complete, then we call H , a right quaternionic Hilbert space.

Throughout this article H denotes a right quaternionic Hilbert space.

If $u, v \in H$, then the following polarization identity [9, Proposition 2.2] holds:

$$(1) \quad 4\langle u | v \rangle = \sum_{l=1, i, j, k} (\|ul + v\|^2 - \|ul - v\|^2)l.$$

Here we give some examples of quaternionic Hilbert spaces.

- (1) Fix $n \in \mathbb{N}$, and define

$$\mathbb{H}^n := \{(q_j)_{j=1}^n : q_l \in \mathbb{H}, l \in \{1, 2, \dots, n\}\}.$$

Then \mathbb{H}^n is a quaternionic Hilbert space with the following operations:

$$(q_j)_{j=1}^n \cdot p = (q_j \cdot p)_{j=1}^n \text{ for } p \in \mathbb{H} \text{ and}$$

$$\langle (p_j)_{j=1}^n | (q_j)_{j=1}^n \rangle = \sum_{j=1}^n \overline{p_j} q_j.$$

- (2) The space of all square summable quaternionic sequences:

$$\ell_{\mathbb{H}}^2(\mathbb{N}) := \{(q_n)_{n=1}^\infty : q_n \in \mathbb{H}, \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n=1}^\infty |q_n|^2 < \infty\},$$

is a quaternionic Hilbert space with the following operations:

$$(q_j)_{j=1}^\infty \cdot p = (q_j \cdot p)_{j=1}^\infty \text{ for } p \in \mathbb{H} \text{ and}$$

$$\langle (p_n)_{n=1}^\infty | (q_n)_{n=1}^\infty \rangle = \sum_{n=1}^\infty \overline{p_n} q_n, \text{ for each } (p_n)_{n=1}^\infty, (q_n)_{n=1}^\infty \in \ell_{\mathbb{H}}^2(\mathbb{N}).$$

Definition 1.1. Let $S \subseteq H$. The orthogonal complement of S is denoted by S^\perp and is defined as

$$S^\perp := \{u \in H \mid \langle u|v \rangle = 0, \text{ for all } v \in S\}.$$

Proposition 1.2. [9, Proposition 2.5] Let \mathcal{N} be a subset of a right quaternionic Hilbert space H such that, for $z, z' \in \mathcal{N}$, $\langle z|z' \rangle = 0$ if $z \neq z'$ and $\langle z|z \rangle = 1$. The following conditions are equivalent:

(1) For every $x, y \in H$, it holds:

$$\langle x|y \rangle = \sum_{z \in \mathcal{N}} \langle x|z \rangle \langle z|y \rangle.$$

The above series converges absolutely

(2) For every $x \in H$, the following identity holds:

$$\|x\|^2 = \sum_{z \in \mathcal{N}} |\langle z|x \rangle|^2$$

(3) $\mathcal{N}^\perp = \{0\}$

(4) $\overline{\text{span}} \mathcal{N} = H$.

Proposition 1.3. [9, Proposition 2.6] Every quaternionic Hilbert space H admits a subset \mathcal{N} , called Hilbert basis of H such that, for $z, z' \in \mathcal{N}$, $\langle z|z' \rangle = 0$ if $z \neq z'$ and $\langle z|z \rangle = 1$, and \mathcal{N} satisfies equivalent conditions stated in proposition 1.2.

Furthermore, if \mathcal{N} is Hilbert basis of H , then every $x \in H$ can be uniquely represented as follows:

$$x = \sum_{z \in \mathcal{N}} z \langle z|x \rangle$$

Definition 1.4. [9, Definition 2.9] A map $T: H \rightarrow H$ is said to be right \mathbb{H} -linear operator, if $T(x \cdot q + y) = T(x) \cdot q + T(y)$, for every $x, y \in H$ and $q \in \mathbb{H}$. Here, T is said to be continuous (bounded), if there exists a $k > 0$ such that $\|Tx\| \leq k\|x\|$, for all $x \in H$. If T is bounded, then

$$\|T\| := \sup\{\|Tx\| : x \in H, \|x\| = 1\},$$

is finite and is called the norm of T .

We denote the set of all right \mathbb{H} -linear operators on H by $\mathcal{B}(H)$.

Definition 1.5. [9] Let $T \in \mathcal{B}(H)$. Then there exists a unique right linear operator denoted by T^* satisfying $\langle x|Ty \rangle = \langle T^*x|y \rangle$, for all $x, y \in H$. This operator T^* is called the adjoint of T .

Definition 1.6. [9] Let $T \in \mathcal{B}(H)$. Then T is said to be

- (1) normal if $T^*T = TT^*$
- (2) self-adjoint if $T^* = T$
- (3) positive if $\langle u|Tu \rangle \geq 0$, for all $u \in H$
- (4) anti self-adjoint if $T^* = -T$
- (5) unitary if $TT^* = T^*T = I$.

Definition 1.7. [9, Section 3.1] Let \mathcal{N} be a Hilbert basis of H . Then the left multiplication induced by \mathcal{N} is a map $\mathcal{L}_{\mathcal{N}}: \mathbb{H} \rightarrow \mathcal{B}(H)$ defined by

$$\mathcal{L}_{\mathcal{N}}(q) = L_q, \text{ for all } q \in \mathbb{H}.$$

Here $L_q(x) = q \cdot x := \sum_{z \in \mathcal{N}} z \cdot q \langle z|x \rangle$, for every $x \in H$.

Lemma 1.8. [9, Lemma 4.1] Let $\langle \cdot | \cdot \rangle : H \times H \rightarrow \mathbb{H}$ be an inner product on H and fix $m \in \mathbb{S}$. Let J be an anti self-adjoint unitary operator on H . Define

$$H_{\pm}^{Jm} = \{x \in H : J(x) = \pm x \cdot m\}.$$

Then

- (1) $H_{\pm}^{Jm} \neq \{0\}$ and the restriction of the inner product $\langle \cdot | \cdot \rangle$ to H_{\pm}^{Jm} is \mathbb{C}_m -valued. Therefore H_{\pm}^{Jm} is \mathbb{C}_m -Hilbert space, called the slice Hilbert space of H
- (2) $H = H_+^{Jm} \oplus H_-^{Jm}$.

Remark 1.9. For $m \in \mathbb{S}$, if \mathcal{N} is a Hilbert basis of H_+^{Jm} , then \mathcal{N} is also a Hilbert basis for H and it holds:

$$J(x) = \sum_{z \in \mathcal{N}} z \cdot m \langle z|x \rangle.$$

That is $J = L_m$, the left multiplication induced by Hilbert basis \mathcal{N} of H (see [9, Proposition 3.8(f)] for details).

2. UNBOUNDED OPERATORS

In this section we prove the spectral theorem for unbounded quaternionic normal operators. First, we restrict the operator to \mathbb{C}_m -linear subspace and apply the classical spectral theorem for unbounded normal operator in a complex Hilbert space. Then, extend the operator by using Proposition 2.7 in this article to establish the result.

Analogous to the complex Hilbert spaces, the theory of unbounded operators in a quaternionic Hilbert space is studied in [9]. We recall some properties of unbounded operators in quaternionic Hilbert space, that we need for our purpose.

Definition 2.1. Let $T: \mathcal{D}(T) \subseteq H \rightarrow H$ be a right linear operator with domain $\mathcal{D}(T)$, a right linear subspace of H . Then T is said to be densely defined, if $\mathcal{D}(T)$ is dense in H .

The graph of an operator T is denoted by $\mathcal{G}(T)$ and is defined as

$$\mathcal{G}(T) = \{(x, Tx) | x \in \mathcal{D}(T)\}.$$

Definition 2.2. Let $T: H \rightarrow H$ be right linear with domain $\mathcal{D}(T) \subseteq H$. Then T is said to be closed, if the graph $\mathcal{G}(T)$ is closed in $H \times H$. Equivalently, if $(x_n) \subset \mathcal{D}(T)$ with $x_n \rightarrow x \in H$ and $Tx_n \rightarrow y$, then $x \in \mathcal{D}(T)$ and $Tx = y$.

Note 2.3. We denote the class of densely defined closed right linear operators in H by $\mathcal{C}(H)$.

Let $S, T \in \mathcal{C}(H)$ with domains $\mathcal{D}(S)$ and $\mathcal{D}(T)$, respectively. Then, S is said to be a restriction of T denoted by $S \subseteq T$, if $\mathcal{D}(S) \subseteq \mathcal{D}(T)$ and $Sx = Tx$, for all $x \in \mathcal{D}(S)$. In this case, T is called an extension of S . We say $S = T$ if $S \subseteq T$ and $T \subseteq S$. In other words, $S = T$ if and only if $\mathcal{D}(S) = \mathcal{D}(T)$ and $Sx = Tx$, for all $x \in \mathcal{D}(T)$.

If $S, T \in \mathcal{C}(H)$ with domains $\mathcal{D}(S)$ and $\mathcal{D}(T)$, respectively, then $\mathcal{D}(ST) = \{x \in \mathcal{D}(T) : Tx \in \mathcal{D}(S)\}$ and $(ST)x = S(Tx)$, for all $x \in \mathcal{D}(ST)$.

Definition 2.4. An operator T with domain $\mathcal{D}(T) \subseteq H$ is said to be normal, if $T \in \mathcal{C}(H)$ and $T^*T = TT^*$.

Definition 2.5. Let $T \in \mathcal{C}(H)$ and $S \in \mathcal{B}(H)$, then we say that S commute with T if $ST \subseteq TS$. In other words, $Sx \in \mathcal{D}(T)$ and $STx = TSx$, for all $x \in \mathcal{D}(T)$.

We recall the notion of the spherical spectrum of a right linear operator in quaternionic Hilbert space.

Spherical spectrum. [9, Definition 4.1] Let $T: \mathcal{D}(T) \rightarrow H$ and $q \in \mathbb{H}$. Define $\Delta_q(T): \mathcal{D}(T^2) \rightarrow H$ by

$$\Delta_q(T) := T^2 - T(q + \bar{q}) + I \cdot |q|^2.$$

The spherical resolvent of T is denoted by $\rho_S(T)$ and is the set of all $q \in \mathbb{H}$ satisfying the following three properties:

- (1) $N(\Delta_q(T)) = \{0\}$
- (2) $R(\Delta_q(T))$ is dense in H
- (3) $\Delta_q(T)^{-1}: R(\Delta_q(T)) \rightarrow \mathcal{D}(T^2)$ is bounded.

Then the spherical spectrum of T is defined by setting $\sigma_S(T) = \mathbb{H} \setminus \rho_S(T)$.

Analogous to the complex spectral measure ([16]), we can define the quaternionic spectral measure as follows:

Definition 2.6. Fix $m \in \mathbb{S}$. Let K be a subset of \mathbb{C}_m and Σ_K be the σ -algebra of K . The quaternionic spectral measure is a map $E: \Sigma_K \rightarrow \mathcal{B}(H)$ such that

- (1) $E(S)$ is an orthogonal projection, for every set $S \in \Sigma_K$
- (2) $E(\emptyset) = 0$ and $E(K) = I$
- (3) If $S_1, S_2 \in \Sigma_K$, then $E(S_1 \cap S_2) = E(S_1) \cdot E(S_2)$
- (4) For $x, y \in H$, the map $E_{x,y}: \Sigma_K \rightarrow \mathbb{H}$ given by

$$E_{x,y}(S) = \langle x | E(S)y \rangle, \text{ for every set } S \in \Sigma_K,$$

is a quaternion valued measure on Σ_K .

When Σ_K is the σ -algebra generated by Borel subsets of a compact or locally compact Hausdorff space, then we require to add one more condition that each $E_{x,y}$ should be regular Borel measure.

Recall that every linear operator in a slice Hilbert space can be extended uniquely to a quaternionic Hilbert space, also the converse is true with some extra condition.

Proposition 2.7. [9, Proposition 3.11] *For every \mathbb{C}_m -linear operator $T: \mathcal{D}(T) \subset H_+^{Jm} \rightarrow H_+^{Jm}$ there exists a unique right \mathbb{H} -linear operator $\tilde{T}: \mathcal{D}(\tilde{T}) \subset H \rightarrow H$ such that $\mathcal{D}(\tilde{T}) \cap H_+^{Jm} = \mathcal{D}(T)$, $J(\mathcal{D}(\tilde{T})) \subset \mathcal{D}(\tilde{T})$ and $\tilde{T}(u) = T(u)$, for every $u \in H_+^{Jm}$. Moreover, the following facts holds:*

- (1) *If $T \in \mathcal{B}(H_+^{Jm})$, then $\tilde{T} \in \mathcal{B}(H)$ and $\|\tilde{T}\| = \|T\|$*
- (2) *$J\tilde{T} = \tilde{T}J$.*

On the other hand, let $V: \mathcal{D}(V) \rightarrow H$ be a right linear operator. Then $V = \tilde{U}$, for some \mathbb{C}_m -linear operator $U: \mathcal{D}(V) \cap H_+^{Jm} \rightarrow H_+^{Jm}$ if and only if $J(\mathcal{D}(V)) \subset \mathcal{D}(V)$ and $JV = VJ$.

Furthermore,

- (1) *If $\overline{\mathcal{D}(T)} = H_+^{Jm}$, then $\overline{\mathcal{D}(\tilde{T})} = H$ and $(\tilde{T})^* = \tilde{T}^*$*
- (2) *If $S: \mathcal{D}(S) \subset H_+^{Jm} \rightarrow H_+^{Jm}$ is \mathbb{C}_m -linear, then $\tilde{S}T = \tilde{S}\tilde{T}$*
- (3) *If S is the inverse of T , then \tilde{S} is the inverse of \tilde{T} .*

In particular, if $T \in \mathcal{B}(H)$ is non self-adjoint, normal, there exist an anti self-adjoint, unitary $J \in \mathcal{B}(H)$ such that $TJ = JT$ (see [9, Theorem 5.9]), where as if $T \in \mathcal{B}(H)$ is self-adjoint, existence of such J is given in [9, Theorem 5.7(b)]. In case, if $T \in \mathcal{C}(H)$ is normal, existence of a anti self-adjoint unitary operator $J \in \mathcal{B}(H)$ such that $JT \subseteq TJ$ is proved in [12, Theorem 5.6]. For the sake of completeness we give the details here.

Theorem 2.8. [12, Theorem 5.6] *Let $T \in \mathcal{C}(H)$ be normal with the domain $\mathcal{D}(T) \subseteq H$. Then there exists an anti self-adjoint, unitary operator J on H such that $JT \subseteq TJ$.*

Proof. Let $\mathcal{Z}_T := T(I + T^*T)^{-\frac{1}{2}}$ be the \mathcal{Z} -transform of T (see [4, Theorem 6.1] for details). Since T is normal, so is \mathcal{Z}_T . Since \mathcal{Z}_T is bounded, by [9, Theorem 5.9], there exist an anti self-adjoint, unitary $J \in \mathcal{B}(H)$ such that $J\mathcal{Z}_T = \mathcal{Z}_T J$ and $J\mathcal{Z}_T^* = \mathcal{Z}_T^* J$. Also J commutes with every bounded operator on H that commutes with $\mathcal{Z}_T - \mathcal{Z}_T^*$. Clearly, J commutes with $(I - \mathcal{Z}_T^* \mathcal{Z}_T)^{-\frac{1}{2}}$. By using the expression of T , that is $T = \mathcal{Z}_T(I - \mathcal{Z}_T^* \mathcal{Z}_T)^{-\frac{1}{2}}$, it can be easily verified that $JT \subseteq TJ$. \square

We recall the spectral theorem for unbounded normal operators in complex Hilbert spaces.

Theorem 2.9. [13, Theorem 13.33] *Every normal operator N in a complex Hilbert space K has a unique spectral decomposition E , which satisfy*

$$\langle a|Nb \rangle = \int_{\sigma(N)} \alpha dE_{a,b}(\alpha), \text{ for all } a \in K, b \in \mathcal{D}(N).$$

Moreover, $ME(\Omega) = E(\Omega)M$, for every subset Ω in the σ - algebra of $\sigma(N)$ and for every bounded operator M on K which commutes with N .

Now we give the spectral representation for unbounded quaternionic normal operators.

Theorem 2.10. *Let $T \in \mathcal{C}(H)$ be normal. Fix $m \in \mathbb{S}$. There exists a Hilbert basis \mathcal{N} of H and a unique quaternionic spectral measure F on the σ - algebra of \mathbb{C}_m^+ such that*

$$\langle x|Ty \rangle = \int_{\sigma_S(T) \cap \mathbb{C}_m^+} \lambda dF_{x,y}(\lambda), \text{ for all } x \in H, y \in \mathcal{D}(T).$$

The above representation of T is established with respect to the Hilbert basis \mathcal{N} .

Furthermore, if $S \in \mathcal{B}(H)$ such that $ST \subseteq TS$ and $ST^ \subseteq T^*S$, then $SF(\Omega) = F(\Omega)S$, for every subset Ω in the σ - algebra of $\sigma_S(T) \cap \mathbb{C}_m^+$.*

Proof. Since T is unbounded quaternionic normal operator, by Theorem 2.8, there exists an anti self-adjoint unitary $J \in \mathcal{B}(H)$ such that T commutes with J . For this fixed $m \in \mathbb{S}$, H_+^{Jm} is \mathbb{C}_m - Hilbert space and it has Hilbert basis, say \mathcal{N} . By Remark 1.9, \mathcal{N} is Hilbert basis for H and $J = L_m$, the left multiplication induced by Hilbert basis \mathcal{N} .

It is clear from Proposition 2.7, that there is a unique \mathbb{C}_m - linear operator in H_+^{Jm} , say $T_+ : \mathcal{D}(T) \cap H_+^{Jm} \rightarrow H_+^{Jm}$ such that $\mathcal{D}(T) = \mathcal{D}(T_+) \oplus \Phi(\mathcal{D}(T_+))$, where $\Phi : H \rightarrow H$ is anti \mathbb{C}_m - linear isomorphism defined by $\Phi(x) = x \cdot n$ (see Proof of [9, Proposition 3.11] for details) and $\widetilde{T_+} = T$. Then T_+ is normal operator. By Theorem 2.9, there exists a unique spectral measure E on the σ - algebra of $\sigma(T_+)$ such that

$$\langle a|T_+b \rangle = \int_{\sigma(T_+)} \lambda dE_{a,b}(\lambda), \text{ for all } b \in \mathcal{D}(T_+), a \in H_+^{Jm}.$$

Define $F(\Omega) = \widetilde{E(\Omega)}$, for every measurable subset Ω of $\sigma(T_+)$. Since $E(\Omega)$ is projection on H_+^{Jm} , the operator $F(\Omega)$ is projection on H , for every Ω . Moreover, Proposition 2.7 implies that F satisfies all the properties listed in Definition 2.6 and it is unique. So F is quaternionic spectral measure.

Let $x, y \in H$. Then $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \in H_+^{Jm}$; $x_2, y_2 \in H_-^{Jm}$. Moreover,

$$F_{x,y}(\Omega) = E_{x_1,y_1}(\Omega) - E_{x_1,y_2 \cdot n}(\Omega) \cdot n + E_{x_2,y_1}(\Omega) - E_{x_2,y_2 \cdot n}(\Omega) \cdot n$$

and

$$\langle x|F(\Omega)y\rangle = \int_{\Omega} dF_{x,y}(\lambda),$$

for any measurable set Ω .

Let $x = x_1 + x_2 \in H$, $y = y_1 + y_2 \in \mathcal{D}(T)$. Then

$$\begin{aligned} \langle x|Ty\rangle &= \langle x_1 + x_2|T(y_1 + y_2)\rangle \\ &= \langle x_1 + x_2|T_+(y_1) - T_+(y_2 \cdot n) \cdot n\rangle \\ &= \langle x_1|T_+(y_1) - T_+(y_2 \cdot n) \cdot n\rangle + \langle x_2|T_+(y_1) - T_+(y_2 \cdot n) \cdot n\rangle \\ &= \int_{\sigma(T_+)} \lambda [dE_{x_1,y_1}(\lambda) - dE_{x_1,y_2 \cdot n}(\lambda) \cdot n + dE_{x_2,y_1}(\lambda) - dE_{x_2,y_2 \cdot n}(\lambda) \cdot n] \\ &= \int_{\sigma(T_+)} \lambda dF_{x,y}(\lambda). \end{aligned}$$

Since $\sigma(T_+) = \sigma_S(T) \cap \mathbb{C}_m^+$ [9], we have

$$\langle x|Ty\rangle = \int_{\sigma_S(T) \cap \mathbb{C}_m^+} \lambda dF_{x,y}(\lambda), \text{ for all } x \in H \text{ and } y \in \mathcal{D}(T).$$

If $S \in \mathcal{B}(H)$ and S commutes with T and T^* , then S commutes with $T - T^*$, so by the construction of J in Theorem 2.8, it is clear that J commutes with S . Thus by Proposition 2.7, there is a unique \mathbb{C}_m -linear operator $S_+ : H_+^{Jm} \rightarrow H_+^{Jm}$ such that $\widetilde{S_+} = S$ and S_+ commutes with T_+ . It follows from Theorem 2.9 that $E(\Omega)S_+ = S_+E(\Omega)$, for every subset Ω in the σ -algebra on $\sigma_S(T) \cap \mathbb{C}_m^+$. Therefore

$$F(\Omega)S = \widetilde{E(\Omega)S_+} = \widetilde{S_+E(\Omega)} = SF(\Omega). \quad \square$$

Theorem 2.11. *Let $T \in \mathcal{B}(H)$ be normal. Fix $m \in \mathbb{S}$. There exists a Hilbert basis \mathcal{N} of H and a unique quaternionic spectral measure F on the σ -algebra generated by Borel subsets of \mathbb{C}_m^+ such that*

$$\langle x|Ty\rangle = \int_{\sigma_S(T) \cap \mathbb{C}_m^+} \lambda dF_{x,y}(\lambda), \text{ for all } x, y \in H.$$

The above representation of T is established with respect to the Hilbert basis \mathcal{N} .

Furthermore, if $S \in \mathcal{B}(H)$ such that $ST = TS$ and $ST^ = T^*S$, then $SF(\Omega) = F(\Omega)S$, for every Borel subset Ω of $\sigma_S(T) \cap \mathbb{C}_m^+$.*

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